Pushdown Automata

A *pushdown automaton*, or PDA, extends the ε-NFA model by adding a stack with its own alphabet Γ (which may be different from Σ). Naturally, only the topmost symbol on the stack is visible.

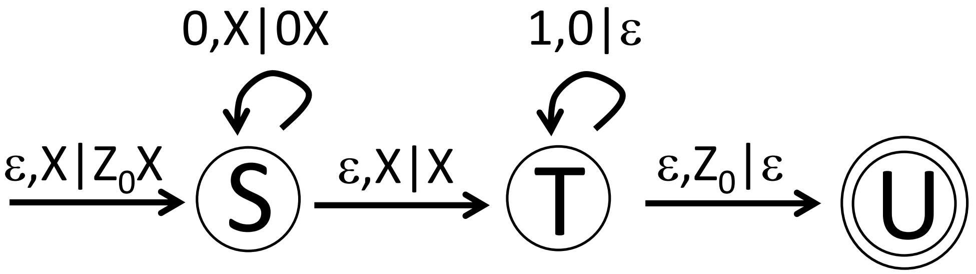
Transition notation for the stack:

$\xrightarrow{a,b|cb}$ means: on input a with b on top of the stack, push c (on top of b).

$\xrightarrow{a,b|c}$ means: on input a with b on top of the stack, pop the stack and push c.

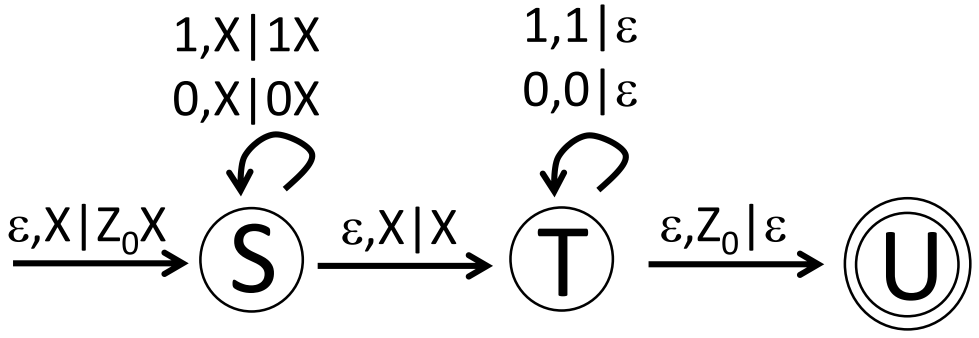
$\xrightarrow{a,b|\varepsilon}$ means: on input a with b on top of the stack, pop the stack.

Here is another example. This automaton accepts $\{0^n1^n | n>=0\}$



We will use the symbol Z_0 for the stack bottom (it marks the empty stack) and X as a placeholder for anything on the stack.

Example: the following PDA accepts strings in (0+1)* that are even-length palindromes



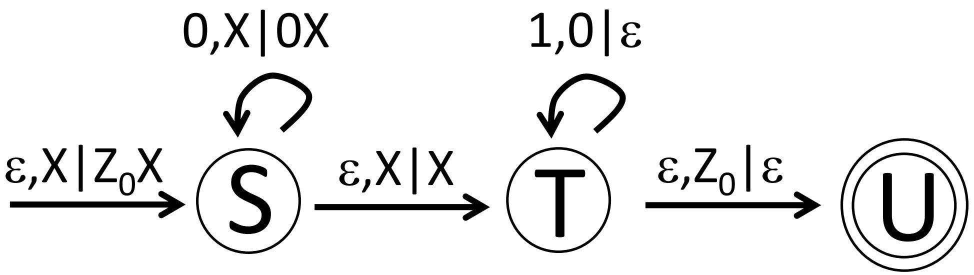
This automaton accepts strings that get to state U after consuming all of their input. Note that if it starts with an empty stack the stack will be empty at the end of the input.

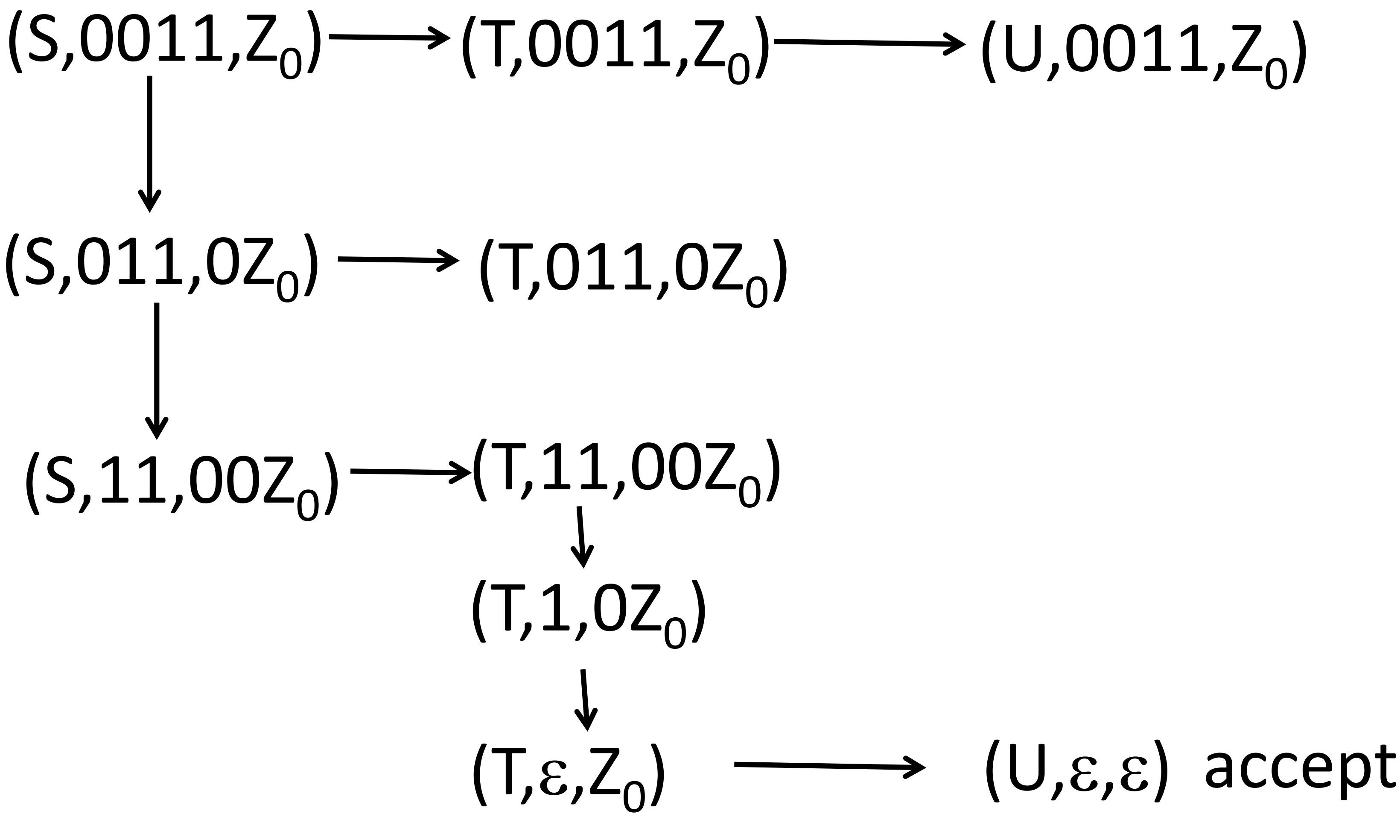
We need a more formal and deterministic way to think about PDA computations.

An *instantaneous description* (ID) is a triple (q, w, γ) where

- q is a state
- w is a string (the portion of the input not yet used)
- g is a string of stack symbols (the complete contents of the stack, with the top on the left)

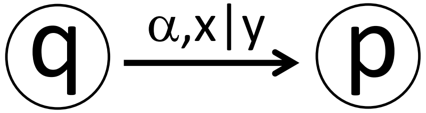
Here is an ID analysis of for the input 0011:





One step in such an ID analysis is $(q,\alpha w,x\beta) \longrightarrow (p,w,y\beta)$

This is valid if the PDA has a transition



We write $(q,w_1,\alpha) \xrightarrow{*} (p,w_2,\beta)$ if there is a sequence of steps that take the PDA from the first ID to the second.

Formally a PDA is a 7-tuple $(\Sigma,Q,\delta,s,F,\Gamma,Z_0)$ where

- Σ,Q,s,F have the same meanings as with DFAs
- δ is our configuration transforomation function
- Γ is the alphabet of stack symbols
- Z_0 is the stack bottom

There are two commonly used definitions of what it means for the PDA to accept a string:

<u>Acceptance by final state</u>: If P is the PDA $(\Sigma,Q,\delta,s,F,\Gamma,Z_0)$ then $\mathcal{F}(P) = \{w \mid \text{there is } q\in F \text{ and } \alpha\in\Gamma^* \text{ so that } (s,w,Z_0) \xrightarrow{*} (q,\varepsilon,\alpha)\}$

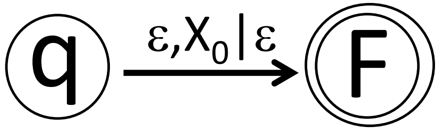
<u>Acceptance by empty stack</u>: If P is the PDA $(\Sigma,Q,\delta,s,F,\Gamma,Z_0)$ then $\mathcal{E}(P) = \{w \mid \text{there is some state } q \text{ so that } (s,w,Z_0) \xrightarrow{*} (q,\varepsilon,\varepsilon)\}$

For a given automaton P, $\mathcal{E}$(P) and $\mathcal{F}$(P) are not necessarily the same. However, the languages that can be accepted by empty stack are the same as those that can be acceped by final state:

Theorem 1: Start with PDA P. Then there is a PDA P' where $\mathcal{F}$(P')=$\mathcal{E}$(P).

Theorem 2: Start with PDA P. Then there is a PDA P' where $\mathcal{E}$(P')=$\mathcal{F}$(P).

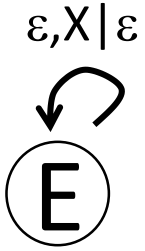
Theorem 1: Start with PDA P. Then there is a PDA P' where $\mathcal{F}(P')=\mathcal{E}(P)$.
Proof: To make P', start with P. Create a new start state s' which pushes a new stack symbol X_0 onto the stack before Z_0. Make a new final state F. For each state q of P add a transition



If P ever accepts a string by emptying its stack, P' can transition to its final state F.
On the other hand, if P' ever accepts a string then at the end of the input there must be only X_0 on the stack, so P must have emptied its stack.

Theorem 2: Start with PDA P. Then there is a PDA P' where $\mathcal{E}$(P')=$\mathcal{F}$(P).

Proof: This is easy. Start with P' the same as P. Give P' a new state E that empties the stack:



and add an ε-transition from every final state to E. String w can take P to a final state if and only if w empties the stack of P'.